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A Revised Identical Elements Model of Arithmetic Fact Representation

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The identical elements model of arithmetic fact representation (T. C. Rickard, A. F. Healy, & L. E. Bourne, 1994) states that, for each triplet of numbers (e.g., *4, 7, 28*) that are related by complementary multiplication and division problems, there are 3 independent fact representations in memory: (*4, 7, x*) → *28*; (*28 / 7*) → *4*; and (*28 / 4*) → *7*. In this article, the author reviews the evidence for this model, considers alternative accounts, and proposes a simple and empirically motivated revision to the model that (a) accommodates conflicting results, (b) provides a novel account of the ties effect, and (c) makes new and nonintuitive predictions for the factoring operation (e.g., factoring of 28 into 4 and 7). The author reports 3 experiments designed to test these predictions and discusses implications for arithmetic instruction.

There is general agreement in the cognitive literature that adult performance on single-digit multiplication and division problems (e.g., *4 × 7* and *28 / 7*) primarily involves fact recall (for reviews, see Ashcraft, 1992, 1995), although other strategies are sometimes used (e.g., LeFevre & Morris, 1999). There is debate, however, on the relation between multiplication and division fact representations. At the broadest level, there are two opposing theoretical frameworks (see Campbell, 1997; LeFevre & Morris, 1999; Rickard, Healy, & Bourne, 1994). According to the *common representations* framework, answers to complementary multiplication and division problems, as in the examples above, are retrieved through a common underlying representation. Among other things, this type of model predicts strong transfer of learning, such that speed-up with practice on a problem like *4 × 7* will transfer to its complementary division problem, *28 / 7*, with analogous transfer from division to multiplication. According to the *independent representations* framework, complementary multiplication and division problems have independent representations, such that practice on one of these problems will not transfer at all to its complementary problem in the other operation. The research to date (Campbell, 1997, 1999; LeFevre & Morris, 1999; Rickard & Bourne, 1996; Rickard et al., 1994) has ruled out a pure common representations model, but consensus has not been reached regarding the detailed characteristics of the independent representations. Hybrid models, which combine properties of both common and separate representations, have also not been ruled out.

The first independent representations model was the identical elements (IE) model (Rickard et al., 1994). For each number triplet in multiplication and division (e.g., *4, 7, 28*), the model specifies three independent facts, each consisting of a representation for the problem and a feed-forward association to the answer, as shown in Figure 1 (ignoring the single left-pointing arrow for now). The central claim of the IE model is that, at high skill levels, there is an independent representation corresponding to each unique combination of (a) the conceptual arithmetic operation to be performed, (b) the numbers presented in the problem statement, and (c) the answer. Within these constraints, the same representation is accessed for retrieval regardless of modality, format, or order of number presentation (temporal or spatial). The model applies to the asymptotic level of organization and representation of arithmetic facts (Rickard & Bourne, 1996, p. 1282) and implicitly allows for the possibility that at some intermediate stage of skill, a different type of fact representation might mediate performance. The higher the level of practice, the greater the likelihood that the representation is asymptotic. Of course, the better the model's account of actual adult performance, the stronger its supporting evidence, and the more useful it is in a practical sense.

Following its initial development in Rickard et al. (1994), the IE model correctly predicted most results from a series of practice-transfer experiments (Campbell, 1997, 1999; LeFevre & Morris, 1999; Rickard & Bourne, 1995, 1996). It also accounts for findings that patients who have a deficit for a subset of multiplication problems exhibit matched deficits for the operand order reversal of those problems (Hittmair-Delazar, Semenza, & Denes, 1994; McCloskey, Aliminosa, & Sokol, 1991).

The only case in which adult performance on single-digit operand problems does not appear to match the asymptotic IE predictions is that of large division problems, for which cross-operation transfer has sometimes been observed (Campbell, 1997, 1999; LeFevre & Morris, 1999). Large problems are surely the least practiced, however (Ashcraft & Christy, 1995; Hanmann & Ashcraft, 1986), so performance on these problems is least likely to reflect direct fact retrieval through asymptotic representations. It appears instead that subjects solve some of these problems using *mediated fact retrieval*. They somehow reframe the division problem (e.g., *56 / 7*) as the corresponding multiplication problem (*8 × 7*) and use the multiplication fact to find the answer (LeFevre & Morris, 1999; Mauro, LeFevre, & Morris, 2003). Indeed, adult subjects have reported using multiplication to do division problems about 37% of the time (LeFevre & Morris, 1999), whereas they have not reported using division facts to mediate multiplica-

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$$(4,7,x) \rightleftarrows 28$$

$$28 \div 4 \rightarrow 7$$

$$28 \div 7 \rightarrow 4$$

Figure 1. The identical elements (right-pointing arrows only) and revised identical elements (all arrows) models. The visual-Arabic representation is used for exposition only and is not implied by the models.

tion. Mediation of large division performance by multiplication could strengthen the multiplication representation, which could in turn yield transfer from division to multiplication on a subsequent test. Likewise, practice on a multiplication problem could facilitate mediated retrieval for the corresponding division problem on a subsequent test. Both types of transfer have been observed for large problems (e.g., Campbell, 1997, 1999; LeFevre & Morris, 1999).

According to the IE model, with sufficient practice, performance on even large division problems should be exclusively mediated by the division representations shown in Figure 1. This account has parsimony, given that (a) performance on multiplication and small division problems is generally as predicted by the model and (b) Rickard and Bourne (1996) found minimal and nonsignificant cross-operation transfer among even large problems following 90 repetitions per problem. Nevertheless, the IE model does not provide an explicit process account of mediated fact retrieval for large division problems prior to laboratory practice.

A Revised (and Reversed) Identical Elements Model

There were four motivations behind the development of the revised identical elements (IE-r) model. The first two motivations were to make the IE framework more consistent with the general memory literature and, in so doing, to provide an explicit account of how multiplication representations might be used to mediate performance on large division problems. A third motivation was to develop a model that, for the first time, makes predictions for factoring (e.g., finding the factors *3* and *7* when presented with *21*).[1] Mathematically speaking, factoring should be considered a third arithmetic operation that is on the same footing as multiplication and division. Factoring skill is also important in a practical sense. Taking a square root is a factoring operation. Children use factoring to perform fractional operations such as reducing to a common denominator. They also encounter the concept of factoring when learning algebra. A final motivation for developing the IE-r model is that it may, if correct, have implications for the improvement of instruction in children's arithmetic.

With regard to the general memory literature, several well-established findings suggest that the IE model, although viable in its general architecture, is incomplete as developed to date. Most pertinently for this article, there is evidence that practice on a paired associate recall item, *a*→*b*, yields substantial (but not 100%) accuracy transfer to its reversed associate, *b*→*a* (see, e.g., Izawa, 1965). Unpublished work in my laboratory generalizes those findings to response times (RTs). Following different levels of *a*–*b* practice, subjects were tested on *b*–*a* associates. Test performance was accurate and was fastest for subjects who had had the most *a*→*b* practice. RTs for the *b*→*a* associates were, however, substantially slower than were RTs for the *a*→*b* associates at the end of *a*→*b* practice. Subjects were not aware that they would be tested on the reverse associates, so formation and strengthening of that association does not appear to require strategic effort.

In light of that work, it appears that a more general and complete version of the IE model (the IE-r model) must include a reverse association for multiplication, as shown by the left-pointing arrow in Figure 1. This reverse association should form concurrently with the forward association for multiplication. It follows that, as children learn their multiplication facts, they are unwittingly accruing associations needed to perform factoring via reverse multiplication, though these associations will be weaker than are the forward associations for multiplication.

With the exception of a few multiplication products (e.g., *24*) that correspond to two sets of factors (e.g., *3, 8* and *4, 6*), this reverse association should allow for retrieval of a unique multiplication problem (i.e., a pair of factors) when a subject is presented with a product. In the IE-r model, it is assumed that similar reverse associations are not, in a behaviorally productive sense, present for division problems, because each division answer is associated with a large number of division problems. A single digit does not constrain the set of possible division problems that could be retrieved enough for it to be a useful retrieval cue. Hence, reverse associations for division are not included in Figure 2.

It is useful to specify two cases of the IE-r model. In its *asymptotic* case, the model makes the same convergence predictions for multiplication and division representation as does the IE model and thus inherits the empirical successes of that model. In its *nonasymptotic* case, the IE-r model provides an explicit account of how multiplication knowledge might be used to mediate performance on division problems. Subjects may solve some large division problems by first performing factoring—by way of the reverse association for multiplication—and then simply finding the missing factor in the presented division problem. For example, when presented with *28 / 7*, subjects could factor 28 into its two single digits, 4 and 7, and then select the missing factor as the answer. I refer to this process as *division by factoring*. According to the IE-r model, this mediating strategy occurs for division only at intermediate skill levels. With further practice, the asymptotic IE-r model should hold.

The nonasymptotic IE-r model can accommodate partial transfer from multiplication to division, from division to multiplication, and between inverse division problems at intermediate skill levels (see the earlier discussion of mediated retrieval transfer effects). In

[1] The term *factoring* as used in this article solely denotes reverse multiplication for single-digit operand problems. So, factoring of 28 in this context refers to finding the two single-digit numbers that, when multiplied together, yield 28. The term does not refer to prime factoring.

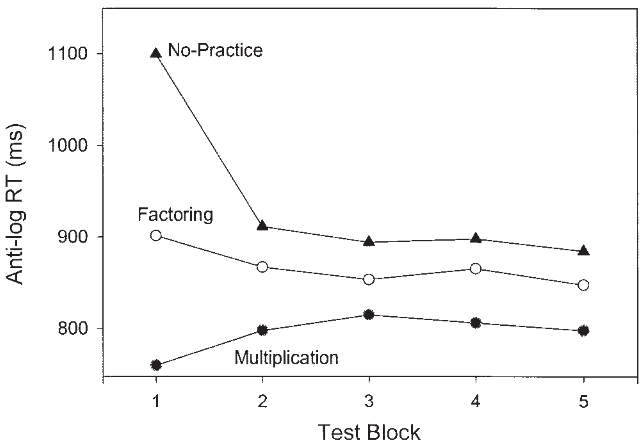

Figure 2. Test response times (RTs) in Experiment 1 as a function of condition and test block.

all cases, it is the common factoring operation in the context of the division-by-factoring strategy that supports the transfer. Therefore, according to the model, if transfer is observed in one of these three directions, it should be observable, given sufficient power, in all three directions. Using an intertrial priming design, Campbell (1999) found RT transfer from multiplication to division but not from division to multiplication. However, LeFevre and Morris (1999) found the predicted bidirectional RT transfer from Block 1 to Block 2 of an alternating-blocks multiplication and division design, although the transfer from division to multiplication was larger than that from multiplication to division.

From the standpoint of the IE-r model, one might ask why division by factoring was apparently not occurring in Rickard and Bourne (1996; Experiment 1). Even if division by factoring is a transitory strategy, replaced by direct retrieval after only a few repetitions, cross-operation transfer effects should have been evident in their transfer data. In fact, closer examination of Rickard and Bourne's Figures 2 and 3 shows that RTs for operation change problems were, for both multiplication and division, slightly faster than those for new (unpracticed) problems, but on the first test block only.

The IE-r model also provides a candidate mechanism for the ties effect, the finding that squares problems (e.g., 4 × 4) are solved faster than adjacent problems on the multiplication table (Ashcraft, 1992, 1995; Campbell, 1999; Campbell & Gunter, 2002). After children learn their multiplication facts (or perhaps during that instruction), they learn to retrieve square roots for the products of ties problems. Retrieving a square root is a special case of factoring that, according to the IE-r model, should strengthen the forward associations for ties multiplication problems (i.e., the bidirectional associative strengthening should operate symmetrically). Hence, when all other variables are held constant, ties multiplication problems should exhibit an RT advantage after children have been taught to do square root calculations. The division-by-factoring hypothesis can similarly account for the ties effect for division (i.e., the finding that a tie problem, like *64 / 8*, is solved faster than nontie division problems with similar dividend magnitude; Campbell & Gunter, 2002). It may be that children and adults routinely use division by factoring for ties division problems because factoring for those problems is well practiced (again by way of solving square root problems). These accounts are not unique in their reliance on differential practice (for a review, see Campbell & Gunter, 2002). However, to date, children's experience retrieving square roots has not been implicated in that practice, and the mechanism elucidated by the IE-r model is unique.

Novel Predictions of the IE-r Model

I tested three predictions of the IE-r model in the experiments reported below. First, the model predicts that educated adults should be able to efficiently factor two-digit multiplication products without prior laboratory task practice. There appears to be no relevant data in the literature. It is conceivable that adults will be able to perform factoring relatively easily through fact retrieval (i.e., reverse multiplication), as the IE-r model implies, but it seems equally likely that they will have to rely on time-consuming backup strategies such as the generate-and-test strategy (i.e., retrieving answers to multiplication problems until the answer matches the presented product). Second, the bidirectional concurrent associative strengthening for multiplication and factoring in the IE-r model predicts that speed-up with practice on factoring should transfer substantially to multiplication, despite the minimal and apparently transitory transfer from division to multiplication (Rickard & Bourne, 1996).

A third new prediction of the IE-r model arises from the 1:2 ratio of representation for factoring versus division. There is a single representation for each number triplet that supports both multiplication and factoring. In contrast, there are two separate division representations for each number triplet (see Figure 1). Thus the IE-r model makes a nonobvious prediction for relative speed-up with practice on division and factoring when considering practice on the complete set of all single-digit problems, or the functional equivalent; although there may be an initial RT advantage for division (among college students) because of greater previous learning for division facts, there should be an interaction such that the rate of speed-up with practice for factoring should exceed that for division. With sufficient practice, the model pre-

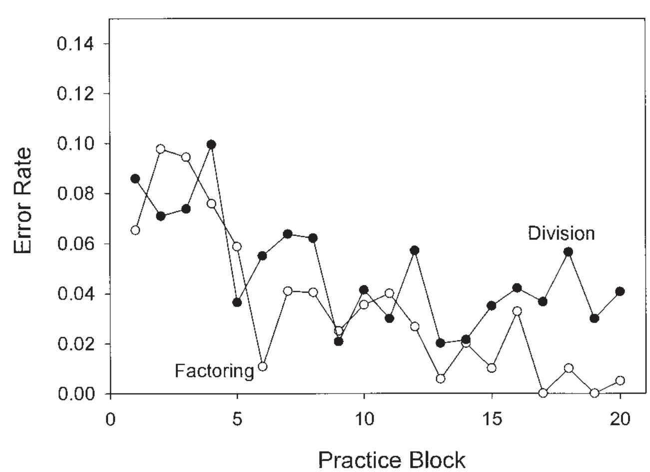

Figure 3. Error rates as a function of task and practice block in Experiment 2.

dicts a crossover interaction, such that factoring speed actually surpasses division speed.

In the factoring tasks of Experiments 1–3, subjects were allowed to speak the factor pair in whichever order they wished. According to the IE-r model, operand order is not encoded in the multiplication representation, so any decision about vocal response order would have to take place postretrieval. Thus, imposition of a factoring response order on subjects (such as smaller operand first) would add an extra decision component to the factoring task that could confound the comparison of relative retrieval difficulty for the two operations that is the goal of Experiments 2 and 3. Further, in a practical or applied sense, the order of factor retrieval and responding is unlikely to be of consequence (see the General Discussion).

Experiment 1

In this experiment, I used a practice-transfer design to test the predictions that adults can factor efficiently and that gains in factoring skill with practice would transfer to multiplication.

Method

Subjects. Twenty-five psychology students at the University of California, San Diego participated for course credit.

Stimuli and apparatus. Multiplication and factoring stimuli were drawn from 24 number triplets (with product values from 12 to 72) divided into three sets of 8 in a manner that roughly equated problem difficulty among the three sets. For each subject, products (e.g., 28) taken from one set of 8 triplets served as stimuli for the factoring task during the practice phase, multiplication problems (6 × 7) taken from another set of triplets served as stimuli for the multiplication task during the practice phase, and the third set of triplets was not presented during the practice phase. During the test phase, all 24 triplets were presented as multiplication problems.

Subjects were tested on IBM-compatible PCs running E-prime software (Psychological Software Tools, Pittsburgh, Pennsylvania). A voice-key apparatus recorded both vocal onset latency (the latency between the onset of the stimulus and the onset of the subject's response) and vocal duration latency.

Design and procedure. During the practice phase, subjects received 20 blocks of multiplication problems interleaved with 20 blocks of factoring problems, with each block consisting of one randomly ordered presentation of each of the 8 arithmetic problems. The role of the 3 problem sets (multiplication, factoring, or unpracticed) was counterbalanced across subjects. There were 5 multiplication blocks in the transfer test phase, each of which contained one randomly ordered presentation of the 24 multiplication problems. For multiplication, operand order was reversed from block to block for each problem during both the practice and test phases.

Subjects were tested in a single 1-hr session. They were instructed to perform each trial quickly and accurately and to try to continually improve their performance. At the end of each practice block, their percent accuracy and mean RT for correctly solved problems was displayed. Each trial began with an asterisk presented at the center of the screen for 500 ms, followed by a blank screen for 500 ms, followed by presentation of the multiplication (e.g., *7 × 4*) or factoring (e.g., *28*) stimulus. For factoring, subjects responded by speaking the multiplication problem (e.g., they would say "four × seven" or "seven × four" if presented with *28*). If the presented product had two pairs of factors, subjects were to say both pairs in whichever order they preferred. The experimenter used the keyboard to enter the subject's response and to indicate whether there was a voice-key failure. No feedback was given on correct trials. On incorrect trials, the visual feedback *incorrect* was centered two lines below the problem, and the correct answer was provided two lines below the *incorrect* statement. The screen then went blank for 500 ms, and the sequence of events described above was repeated for the next trial. Trial events on the test were identical to those of the practice phase.

Results and Discussion

On the first practice block, the accuracy rate for multiplication was .96 and that for factoring was .895. Here and elsewhere, a factoring trial was scored as an error if either or both retrieved factors were incorrect. On the second block, factoring accuracy increased to .958. Factoring and multiplication accuracy were roughly the same beyond the second block, and on the last block the accuracy rate for both tasks was .985.

Voice-key errors occurred on 4.9% of the practice trials. Here and elsewhere, these trials are not analyzed further. I did not include 14 practice trials with vocal onset RTs < 300 ms in the RT analyses, which were conducted on correct trials only (the same filtering was used for all subsequent RT analyses). There was substantial speed-up during practice. The antilog of the mean log vocal onset RT (averaged over trials within block and then over subjects) for multiplication decreased by 303 ms, from 983 ms on the first practice block.[2] For the factoring task, there was a decrease of 654 ms, from 1461 ms on the first practice block.

Averaged over the five test blocks, multiplication error rates were .023, .038, and .055 for the multiplication practice, factoring practice, and no-practice conditions, respectively. The antilog vocal onset RTs on the transfer test are shown in Figure 2. The condition labels represent the status of the problems during practice. At test, all items were multiplication problems. The results show clear and substantial transfer from factoring to multiplication throughout the test. On the first test block, where pure transfer can be measured without contamination from subsequent multiplication practice effects, about 59% of the improvement obtained through multiplication practice was obtained through factoring practice. A 2 (condition: factoring vs. no practice) × 5 (test block) within-subjects analysis of variance was performed on the subject mean log RTs to test for significance of the factoring transfer effect. There were significant effects of condition, $F(1, 24) = 11.65$, $p = .0023$; test block, $F(4, 96) = 13.56$, $p < .0001$; and their interaction, $F(4, 96) = 7.01$, $p < .0001$.

Reverse associations from the products to the multiplication representations apparently did exist prior to the experiment and were strengthened during factoring practice. Further, this strengthening transferred substantially to multiplication, as the IE-r model predicts. This result demonstrates the necessity of a reverse association for multiplication and of concurrent bidirectional associative strengthening for multiplication and factoring in any complete model of cognitive arithmetic.

Experiment 2

Subjects were given practice on alternating blocks of 8 division and 8 factoring problems to determine whether, as predicted by the IE-r model, the rate of speed-up with practice for factoring is faster

[2] Log-transformed RTs tend to pull in outliers and yield more powerful tests. All RT analyses were performed on the log RTs, although the patterns of significance held when analyses were performed on the raw RTs.

than that for division. To simulate the entire universe of single-digit operand items, the presented division problems were inverted from block to block. Thus, if *28 / 7* was presented on the first division block, *28 / 4* was presented on the second division block, and so on. In contrast, the same 10 factoring problems were presented on each factoring block. This arrangement preserves the 1:2 ratio of factoring to division problems and representations that, according to the IE-r model, exists for the universe of all single-digit operand problems (with the exception of ties problems). Thus, it is an appropriate test of the relative difficulty of the two operations outside of the laboratory.

The literature review suggests that problem size may be an important factor in the outcome. For small division problems, representations prior to the practice phase are most likely to be consistent with the asymptotic IE-r model, so that subjects perform inverse division problems by way of independent division representations. Hence, each division problem receives 10 practice trials over the 20 practice blocks and its inverse receives 10 practice trials over the 20 practice blocks. Each factoring problem, on the other hand, receives 20 practice trials over the 20 practice blocks. Under these conditions, the IE-r model makes the straightforward prediction that the rate of speed-up from block to block should be faster for factoring than it is for division, simply because factoring items get twice as much practice.

A second prediction for small problems is that there should be no speed-up in division RTs from each odd-numbered to the next even-numbered practice block. According to the IE-r model, each division problem and its inverse are solved through completely independent representations, so that practice on *21 / 3* cannot produce any positive transfer to *21 / 7* when presented on the next practice block.

For large problems, it is less clear what forms of representation and what strategies will be involved for division. Subjects will likely have asymptotic IE-r representations for some division problems but not for others. The speed-up advantage for factoring may thus be attenuated relative to that for small problems.

Method

Subjects. Twenty-five psychology students at the University of California, San Diego participated for course credit.

Materials, apparatus, design, and procedure. Stimuli were derived from 16 number triplets having products ranging from 10 to 72. The apparatus was identical to that used in Experiment 1. There were 40 practice blocks alternating between sets of 8 factoring trials and sets of 8 division trials.[3] Number triplets presented for factoring and division were counterbalanced over subjects. For half of the subjects, the first practice block involved division. Trial-level events were identical to those of Experiment 1.

Results and Discussion

Mean error rates are shown in Figure 3 as a function of task and practice block. An advantage for factoring was clear by the end of practice. Averaged over the last five practice blocks, 17 of the 20 subjects who made any errors had a higher error rate for division (sign test, $p < .01$).

The problem size distinction was determined by first rank ordering the 16 nonties products (dividends for division problems) according to their grand mean vocal onset RTs, averaged over blocks, operation (factoring and division), and then subjects. The 8 products (dividends) with slower RTs (63, 28, 72, 32, 42, 56, 54, and 48) were labeled as large problems for both the factoring and division task, and the 8 products (dividends) with the faster RTs (10, 21, 14, 20, 30, 35, 40, and 45) were labeled as small problems for both factoring and division tasks.

The means of the log-transformed vocal onset RTs are shown in Figure 4 (Panels A and B represent small and large problems, respectively). Rickard et al. (1994) and Rickard and Bourne (1996) showed that speed-up in the average RTs for single-digit arithmetic is roughly linear when data are plotted in log–log coordinates. Therefore, least squares linear regression fits for each task are also shown. The results are as predicted by the IE-r model; both the data and the fits reveal a steeper slope for factoring than for division, as well as a crossover interaction.

General linear model analyses were performed on the log RT data shown in Figure 4, with a continuous linear factor of log block, a categorical factor of task, and the interaction. Here and in Experiment 3, the effect of log block was highly significant (Fs > 300). Of more interest, the interaction term was significant for small problems, $F(1, 36) = 7.85$, $p = .008$, and approached significance for large problems, $F(1, 36) = 2.94$, $p = .095$. The effect of task, which in this analysis is a test for an RT effect on the first practice block, was not significant either for small problems, $F(1, 36) = 1.90$, $p = .177$, or for large problems, $F(1, 36) < 1.0$. The effect of task on the last practice block was evaluated by subtracting 1.301 (the log value for the 20th practice block) from the log block variable, which set the value of log block on the 20th block to zero. The test result was highly significant for both small problems, $F(1, 36) = 18.0$, $p < .0001$, and large problems, $F(1, 36) = 6.0$, $p = .019$. This pattern replicates earlier results for 11 subjects who were run in a pilot version of the task.

It is striking that the RT crossover occurred on about the third practice block. This effect shows that the learning advantage for factoring in this design was quite potent, overwhelming any previous learning advantage for division with only minimal practice. Although the RT advantage for division on the first practice block did not approach significance, there is reason to suspect that division performance should be faster initially, so the crossover effect evident in Figure 4 may be a real attribute of the population. The results of Experiment 3 will buttress this conclusion.

As predicted by the IE-r model, solving a division problem did not yield positive transfer to its inverse problem. In a comparison of small problem division RTs in Figure 4 across the 10 pairs of adjacent practice blocks (1 and 2, 3 and 4, 5 and 6, etc.), division performance was actually a bit slower on the even-numbered blocks in 9 of 10 cases (sign test, $p = .01$). In contrast, factoring performance was faster on even-numbered blocks in 7 of 10 cases. These results suggest that solving a division problem may actually interfere with solving its inverse on subsequent trials. See Rickard and Bourne (1996) for a discussion of similar interference effects for inverse division problems as indexed by error patterns. Although the IE-r model does not accommodate interference (i.e.,

[3] There were actually two additional problems per block for each operation. By error, however, these were all ties problems. Because inverse division problems are not defined for ties, data for these items were not analyzed.

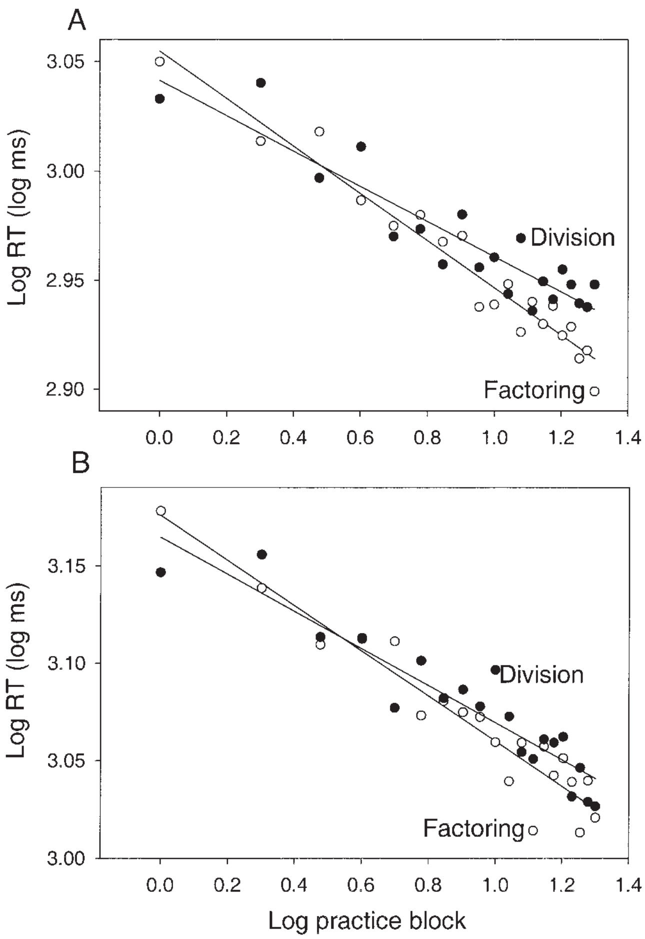

Figure 4. Practice response times (RTs) in Experiment 2 as a function of task and practice block. Panel A shows results for small problems, and Panel B shows results for large problems.

negative transfer) phenomena, the existence of interference seems sensible only if inverse division problems have separate representations. A similar trend was obtained for large problems (division: RTs on even-numbered blocks were slower in 7 of 10 cases; factoring: RTs on even-numbered blocks were faster in 7 of 10 cases).

The different response requirements for the two tasks complicate interpretation somewhat. The division task required subjects to speak a single number into the microphone, whereas the factoring task required subjects to speak a multiplication problem (e.g., "four × seven") into the microphone. The factoring responses thus had longer mean vocalization duration (667 ms as measured by the difference between the onset and offset latencies) than did the division responses (262 ms). It is possible that, instead of fully retrieving both factors prior to response initiation, subjects retrieved one factor and initiated the vocal response for that factor while simultaneously retrieving the second response from memory. This scenario could result in an underestimate of true factoring latency in the vocal onset RTs, leading to the conclusion that factoring was the faster performed retrieval task, when in fact it was not.

Several aspects of the design and data speak against that possibility, however. First, subjects were instructed not to initiate their vocal responses until the answer had been fully retrieved. Second, error rates were significantly lower for the factoring task by the end of practice, a result that is insensitive to response effects in the RT data. Third, if subjects were executing part of retrieval for the factoring task during their vocal response, one would expect that tendency to decrease with practice, as their memory retrieval became more efficient and automatic. This would likely yield decreasing vocal response duration for the factoring task as a function of practice. However, regression analyses in which practice block was used to predict vocal response duration revealed no trend toward decreasing vocal duration RTs during practice for either task. Fourth, the hypothesis that subjects finished their answer retrieval for factoring during the vocal response does not predict the Task × Practice Block crossover interaction that was observed in the RTs. Finally and most conclusively, the results of Experiment 3 confirm that the results of this experiment cannot be an artifact of the different vocal response requirements.

Experiment 3

I designed this experiment to confirm that the results of Experiment 2 were due to the 1:2 representational property of the IE-r model for factoring versus division. The basic design is identical to that of Experiment 2, with the exception that inverse division problems were not presented. If, for example, *28 / 4* was presented on the first practice block, it was also presented on every subsequent division practice block, and *28 / 7* was never presented. This simple change has important consequences from the standpoint of the model. Because the 1:2 ratio of factoring to division problems in Experiment 2 (in terms of the underlying representation according to the IE-r model) is changed to a 1:1 ratio in this experiment, the advantage in rate of speed-up for factoring should be eliminated. Instead, the previous learning advantage for division that is implied by first block performance in Experiment 2 should be evident throughout practice. Alternatively, the results of Experiment 2 might be due to extraneous factors such as the different perceptual characteristics and different response requirements of the two tasks. In this second case, the results of this experiment should mirror the results of Experiment 2, because no changes have been made for those aspects of the tasks.

Method

Subjects. Twenty-four psychology students at the University of California, San Diego, participated for course credit.

Materials, apparatus, design, and procedures. The experiment was identical to Experiment 2 with two exceptions. First, inverse division problems were not presented. Second, 30 practice blocks were given for each task, as opposed to 20 in Experiment 2.

Results and Discussion

Mean error rates are shown in Figure 5 as a function of task and practice block. Throughout practice, error rates were slightly higher for the factoring task. In this experiment, small problems (identified using the same method as for Experiment 2) corresponded to the products (dividends) 10, 21, 14, 20, 25, 30, 35, 32,

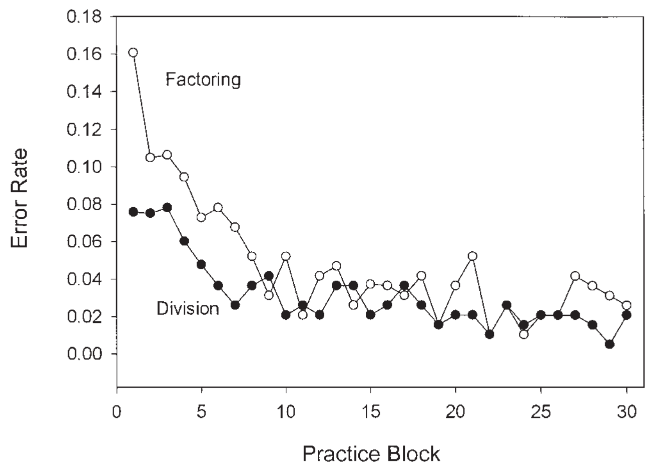

Figure 5. Error rates as a function of task and practice block in Experiment 3.

and 45, and large problems corresponded to the products (dividends) 42, 63, 28, 40, 72, 56, 54, and 48.

The mean log vocal onset RTs for correctly solved small and large problems are shown in Figure 6 (Panels A and B, respectively). Throughout practice, division was performed faster than factoring. In the same general linear model analysis as performed for Experiment 2, the Task × Log Block interaction was not significant for either small problems, $F(1, 59) < 1.0$, or large problems, $F(1, 59) < 1.0$. There was no effect of task on the first practice block for small problems, $F(1, 59) = 1.05$, but there was a significant effect for large problems, $F(1, 59) = 4.25$, $p = .044$. There was an overwhelming statistical advantage for division on the last practice block for both small, $F(1, 59) = 19.19$, $p < .0001$, and large, $F(1, 59) = 18.72$, $p < .0001$, problems. Comparison of division RTs on odd- and even-numbered blocks showed speed-up on 10 of the 15 block pairs for small problems and on 11 of the 15 block pairs for large problems, as expected given the design change in this experiment.

This experiment was identical to Experiment 2 in nearly every respect; the only relevant exception was that inverse division problems were not presented. Yet division was the faster operation throughout practice, and there were no interactions involving task and practice block. Thus, this experiment rules out any extraneous variables, including the different perceptual properties or response requirements for the factoring and division tasks, as possible accounts of the results of Experiment 2.

General Discussion

The IE-r model provides a new reference point for research into adults' mental organization arithmetic facts. It predicts or accommodates nearly all transfer results for single-digit multiplication and division, in some case uniquely. I tested and confirmed several of its claims about factoring here, including the predictions that (a) educated adults are able to factor two-digit multiplication products with relative ease, apparently using a reverse multiplication retrieval process; (b) improvements in performance through factoring practice transfer substantially to multiplication; and (c) factoring is performed more quickly than division, given only a few practice trials and problem sets that mimic the real world.

As Rickard and Bourne (1996) noted, the IE model also makes predictions for adult addition and subtraction performance that have yet to be tested. Asymptotically, the transfer predictions for addition and subtraction are identical to the predictions for multiplication and division, respectively. However, factoring has no analog in the case of addition. That is, because each addition answer is the sum of numerous different operand combinations, there should be no behaviorally productive reverse associations for addition (just as for division and subtraction). For addition and subtraction, then, the IE and IE-r models make identical predictions.

Despite its successes, the IE-r model, with its focus on fact representation and consequent positive transfer, explains only one aspect of adult arithmetic performance. Other common phenomena include strategy shifts with practice (Logan, 1988; Reder & Ritter, 1992; Rickard, 1997, 2004; Siegler, 1988) as well as cross-problem interference within fact retrieval itself, due, for example, to operand overlap among problems (for review, see Ashcraft, 1995). To account for these interference effects, any candidate

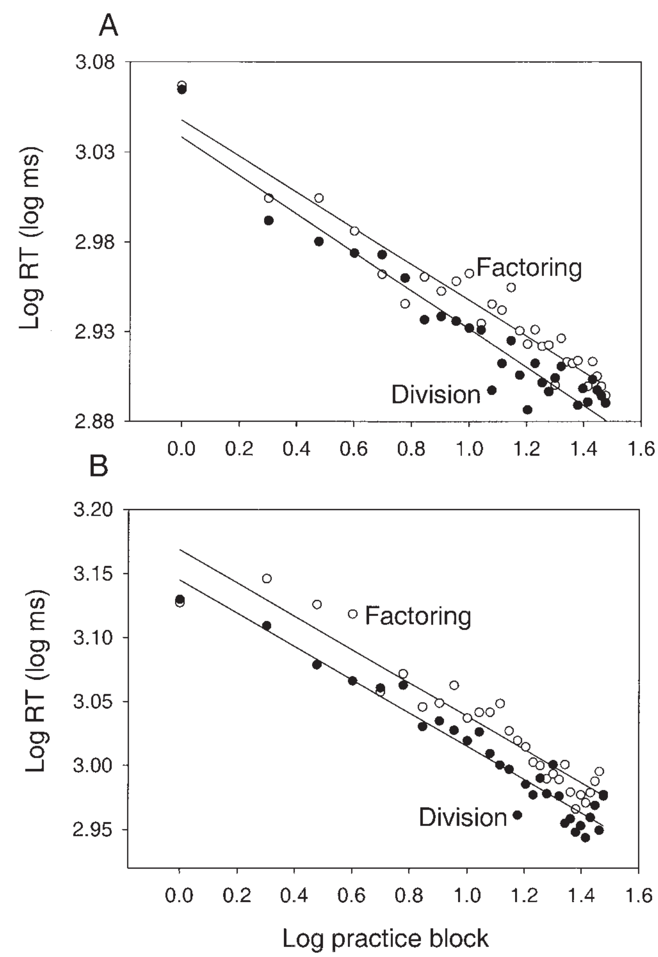

Figure 6. Practice response times (RTs) in Experiment 3 as a function of task and practice block. Panel A shows results for small problems, and Panel B shows results for large problems.

model must specify how various arithmetic facts are related within an associative network. Integration of the IE-r model into such a network remains an important goal for future work.

Research that explores the possible role of multiplication knowledge in division performance is of particular interest because of its potential application to children's division learning. Division is apparently the more difficult operation to commit to memory, and children often fail to master this skill by the time formal instruction on the topic ends. The finding that multiplication knowledge is used even by educated adults to assist in retrieval of large division facts implies that children may employ a similar strategy or could benefit from explicit instruction on the strategy. According to the IE-r model, the ability to factor should develop automatically and concurrently with learning of the multiplication facts, so, after learning their multiplication facts, children should have a head start for performing factoring and division by factoring, relative to performing direct division fact retrieval. The IE-r model further suggests that explicit drill on factoring, either following or concurrent with drill on multiplication, may facilitate the speed with which children can bootstrap into fast and accurate division performance. According to the model, mastering the entire set of single-digit operand problems requires encoding half as many factoring facts as it does division facts, so the rate of learning should be much faster for factoring (as in Experiment 2). Such drill would also make factoring readily available when students begin doing fractional operations. Finally, the finding of Experiment 1 that factoring transfers substantially to multiplication indicates that drill on factoring might have the dual benefit of facilitating children's performance on multiplication as well as division.

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